

Foundations and Measures of Quantum Non-Markovianity

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Abstract

The basic features of the dynamics of open quantum systems, such as the dissipation of energy, the decay of coherences, the relaxation to an equilibrium or non-equilibrium stationary state, and the transport of excitations in complex structures are of central importance in many applications of quantum mechanics. The theoretical description, analysis and control of non-Markovian quantum processes play an important role in this context. While in a Markovian process an open system irretrievably loses information to its surroundings, non-Markovian processes feature a flow of information from the environment back to the open system, which implies the presence of memory effects and represents the key property of non-Markovian quantum behavior. Here, we review recent ideas developing a general mathematical definition for non-Markovianity in the quantum regime and a measure for the degree of memory effects in the dynamics of open systems which are based on the exchange of information between system and environment. We further study the dynamical effects induced by the presence of system-environment correlations in the total initial state and design suitable methods to detect such correlations through local measurements on the open system.

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I. INTRODUCTION

The standard approach to the theoretical description and analysis of dissipation and decoherence processes in open quantum systems presupposes a time evolution without memory. Employing the concept of a quantum Markov process which is given by a semigroup of completely positive dynamical maps one obtains a quantum Markovian master equation describing the time evolution of the reduced open system states with a generator in Lindblad form [1, 2]. Within a microscopic approach quantum Markovian master equations are usually obtained by means of the Born-Markov approximation which assumes a weak system-environment coupling and several further, mostly rather drastic approximations. However, in many processes occurring in nature these approximations are not applicable, a situation which occurs, in particular, in the cases of strong system-environment couplings, structured and finite reservoirs, low temperatures, as well as in the presence of large initial system-environment correlations [3]. In the case of substantial quantitative and qualitative deviations from the dynamics of a quantum Markov process one often speaks of a non-Markovian process, implying that the dynamics is governed by significant memory effects. However, a consistent general theory of quantum non-Markovianity does not exist and even the very definition of non-Markovianity is a highly topical issue. Very recently important steps towards a general theory of non-Markovianity have been made which try to rigorously define the border between Markovian and non-Markovian quantum dynamics and to develop quantitative measures for the degree of memory effects in open systems [4–6]. These approaches provide general mathematical characterizations of quantum non-Markovianity which are independent from any specific representation or approximation of the dynamics, e.g. in terms of a perturbative master equation.

The key questions to be studied here are, how can one mathematically define and quantify non-Markovian behavior in the quantum regime, how do quantum memory effects manifest themselves in the dynamical behavior of complex open quantum systems, and how can such effects be uniquely identified? The answer to these questions is of great relevance for the design of appropriate schemes allowing the experimental detection and measurement of non-Markovianity. Here, we review some recent ideas and concepts which characterize non-Markovian quantum behavior by means of the information which is exchanged between an open quantum system and its environment [5, 7]. The gain or loss of information can

be quantified through the dynamics of the trace distance between a pair of quantum states of the open system. It is known that this distance measure for quantum states can be interpreted as a measure for the distinguishability of the states [8–10]. Markovian processes tend to continuously reduce the trace distance and, hence, the distinguishability between any pair of physical states, which means that there is a flow of information from the open system to its environment. In view of this interpretation the characteristic feature of a non-Markovian quantum process is the increase of the distinguishability, i.e., a reversed flow of information from the environment back to the open system. Memory effects thus emerge through a recycling of information such that earlier states of the open system influence its future states.

We first recapitulate in Sec. II all necessary concepts from the quantum theory of open systems, such as completely positive quantum dynamical maps, dynamical semigroups, Lindblad generators, quantum master equations and the notion of the divisibility of dynamical maps. The precise definition of non-Markovian quantum dynamics and the corresponding measure for the degree of memory effects will then be discussed in detail in Sec. III. This section also contains a summary of the most important mathematical and physical features of the trace distance. In Sec. IV we study a further important aspect, namely the dynamical influence of correlations in the initial system-environment state. We will derive general inequalities expressing this influence and develop experimentally realizable schemes to locally detect such correlations. Finally, some conclusions and further developments will be discussed in Sec. V.

II. OPEN QUANTUM SYSTEMS: BASIC NOTIONS AND CONCEPTS

We introduce and discuss some of the most important notions and general concepts of the quantum theory of open systems. The presentation is of course not an exhaustive review, but we concentrate on those topics that are needed for the discussions of quantum non-Markovianity in the following sections.

A. Microscopic representation of open systems

An open quantum system S is a quantum system which is coupled to another quantum system E , its environment. Thus, S can be regarded as subsystem of the total system $S + E$ consisting of open system plus environment. Denoting the Hilbert spaces of S and E by \mathcal{H}_S and \mathcal{H}_E , respectively, the Hilbert space of the total system is given by the tensor product space

$$\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_E. \quad (1)$$

The physical states of the total system are described by density matrices ρ , representing positive trace class operators on \mathcal{H} with unit trace. This means that $\rho \geq 0$, which implies that ρ is self-adjoint with nonnegative eigenvalues, and $\text{tr} \rho = 1$. The corresponding reduced states of subsystems S and E are given by partial traces over \mathcal{H}_E and \mathcal{H}_S , respectively,

$$\rho_S = \text{tr}_E \rho, \quad \rho_E = \text{tr}_S \rho. \quad (2)$$

In the following we denote the convex set of physical states pertaining to some Hilbert space \mathcal{H} by $S(\mathcal{H})$.

We will assume here that the total system $S + E$ is closed and follows a unitary dynamics described by some unitary time evolution operator

$$U(t) = \exp[-iHt] \quad (\hbar = 1) \quad (3)$$

with a Hamiltonian of the most general form:

$$H = H_S \otimes I_E + I_S \otimes H_E + H_I, \quad (4)$$

where H_S and H_E are the self-Hamiltonians of system and environment, respectively, and H_I is any interaction Hamiltonian. Thus, the time dependence of the total system states is given by the von Neumann equation,

$$\frac{d}{dt} \rho(t) = -i[H, \rho(t)], \quad (5)$$

with the formal solution

$$\rho(t) = U(t) \rho(0) U^\dagger(t). \quad (6)$$

It turns out that in most cases of practical relevance a full solution of the equations of motion on the microscopic level is impossible. Thus, one of the central goals of the quantum

theory of open systems [3] is the development of an effective and analytically or numerically feasible formulation of the dynamical behavior of a suitably defined reduced set of variables forming the open subsystem S . Given a certain split of the total system into open system S and environment E , one tries to derive effective equations of motion for the time dependence of the reduced system state $\rho_S(t)$ through an elimination of the environmental variables from the dynamical equations. The main aim is to develop efficient descriptions for a wide class of physical problems and phenomena, such as the dissipation and damping of populations, the relaxation to a thermal equilibrium state, the emergence of non-equilibrium stationary states, the suppression or destruction of quantum coherences, the description of the quantum transport of excitations in complex systems, and the dynamics of quantum correlations and entanglement.

B. Quantum dynamical maps

Quantum dynamical maps represent a key concept in the theory of open systems. To introduce this concept we presuppose (i) that the dynamics of the total system is given by a unitary time evolution (6), and (ii) that system S and environment E are statistically independent at the initial time, i.e., that the initial total system state represents a tensor product state

$$\rho(0) = \rho_S(0) \otimes \rho_E(0). \quad (7)$$

While condition (i) can be relaxed to include more general cases, condition (ii) is crucial for the following considerations. We will discuss in Sec. IV the case of initially correlated system-environment states. On the basis of these assumptions the open system state at time $t \geq 0$ can be written as follows,

$$\rho_S(t) = \text{tr}_E \{ U(t) \rho_S(0) \otimes \rho_E(0) U^\dagger(t) \}. \quad (8)$$

In the following we consider the initial environmental state $\rho_E(0)$ to be fixed. For each fixed $t \geq 0$ Eq. (8) is then seen to represent a linear map

$$\Phi(t, 0) : S(\mathcal{H}_S) \longrightarrow S(\mathcal{H}_S) \quad (9)$$

on the open system's state space $S(\mathcal{H}_S)$ which maps any initial open system state to the corresponding open system state at time t :

$$\rho_S(0) \mapsto \rho_S(t) = \Phi(t, 0) \rho_S(0). \quad (10)$$

This is the quantum dynamical map corresponding to time t . It is easy to check that it preserves the Hermiticity and the trace of operators, i.e., we have

$$\text{tr}_S \{ \Phi(t, 0) A \} = \text{tr}_S \{ A \}, \quad (11)$$

and

$$[\Phi(t, 0) A]^\dagger = \Phi(t, 0) A^\dagger. \quad (12)$$

Moreover, $\Phi(t, 0)$ is a positive map, i.e., it maps positive operators to positive operators,

$$A \geq 0 \implies \Phi(t, 0) A \geq 0. \quad (13)$$

An important further property of dynamical maps is that they are not only positive but also completely positive. Such maps are also known as trace preserving quantum operations or quantum channels in quantum information and communication theory [11]. We recall that a linear map Φ is completely positive if and only if it admits a Kraus representation [12], i.e., if there are operators Ω_i on the underlying Hilbert space \mathcal{H}_S such that

$$\Phi A = \sum_i \Omega_i A \Omega_i^\dagger. \quad (14)$$

Such a map is trace preserving if and only if the normalization condition

$$\sum_i \Omega_i^\dagger \Omega_i = I_S \quad (15)$$

holds. The original definition of complete positivity, which is equivalent to the existence of a Kraus representation, is the following. Consider for any $n = 1, 2, \dots$ the tensor product $\mathcal{H}_S \otimes \mathbb{C}^n$, describing the Hilbert space of S combined with an n -level system, and the corresponding linear tensor extension of Φ defined by $(\Phi \otimes I_n)(A \otimes B) = (\Phi A) \otimes B$. The map $\Phi \otimes I_n$ thus describes an operation which acts only on the first factor of the composite system, and leaves unchanged the second factor. The map Φ is then defined to be n -positive if $\Phi \otimes I_n$ is a positive map, and completely positive if $\Phi \otimes I_n$ is a positive map for all n . We note that positivity is equivalent to 1-positivity, and that for a Hilbert space with finite dimension $N_S = \dim \mathcal{H}_S$ complete positivity is equivalent to N_S -positivity [13].

If we now allow the time parameter t to vary (keeping fixed the initial environmental state $\rho_E(0)$), we get a one-parameter family of dynamical maps,

$$\{ \Phi(t, 0) \mid t \geq 0, \Phi(0, 0) = I \}, \quad (16)$$

which contains the complete information on the dynamical evolution of all possible initial system states. Thus, formally speaking a quantum process of an open system is given by such a one-parameter family of completely positive and trace preserving (CPT) quantum dynamical maps.

As a simple example we consider the decay of a two-state system into a bosonic reservoir [3, 14]. The total Hamiltonian of the model is given by Eq. (4) with the system Hamiltonian

$$H_S = \omega_0 \sigma_+ \sigma_-, \quad (17)$$

describing a two-state system (qubit) with ground state $|0\rangle$, excited state $|1\rangle$ and transition frequency ω_0 , where $\sigma_+ = |1\rangle\langle 0|$ and $\sigma_- = |0\rangle\langle 1|$ are the raising and lowering operators of the qubit. The Hamiltonian of the environment,

$$H_E = \sum_k \omega_k b_k^\dagger b_k, \quad (18)$$

represents a reservoir of harmonic oscillators with creation and annihilation operators b_k^\dagger and b_k satisfying Bosonic commutation relations $[b_k, b_{k'}^\dagger] = \delta_{kk'}$. The interaction Hamiltonian takes the form

$$H_I = \sum_k \left(g_k \sigma_+ \otimes b_k + g_k^* \sigma_- \otimes b_k^\dagger \right). \quad (19)$$

The model thus describes for example the coupling of the qubit to a reservoir of electromagnetic field modes labelled by the index k , with corresponding frequencies ω_k and coupling constants g_k . Since we are using the rotating wave approximation in the interaction Hamiltonian, the total number of excitations in the system,

$$N = \sigma_+ \sigma_- + \sum_k b_k^\dagger b_k, \quad (20)$$

is a conserved quantity. The model therefore allows to derive an analytical expression for the dynamical map (10). Assuming the environment to be in the vacuum state $|0\rangle$ initially one finds:

$$\rho_{11}(t) = |G(t)|^2 \rho_{11}(0), \quad (21)$$

$$\rho_{00}(t) = \rho_{00}(0) + (1 - |G(t)|^2) \rho_{11}(0), \quad (22)$$

$$\rho_{10}(t) = G(t) \rho_{10}(0), \quad (23)$$

$$\rho_{01}(t) = G^*(t) \rho_{01}(0), \quad (24)$$

where the $\rho_{ij}(t) = \langle i | \rho_S(t) | j \rangle$ denote the matrix elements of $\rho_S(t)$. The function $G(t)$ introduced here is defined to be the solution of the integro-differential equation

$$\frac{d}{dt}G(t) = - \int_0^t dt_1 f(t-t_1)G(t_1) \quad (25)$$

corresponding to the initial condition $G(0) = 1$, where the kernel $f(t-t_1)$ represents a certain two-point correlation function,

$$\begin{aligned} f(t-t_1) &= \langle 0 | B(t) B^\dagger(t_1) | 0 \rangle e^{i\omega_0(t-t_1)} \\ &= \sum_k |g_k|^2 e^{i(\omega_0 - \omega_k)(t-t_1)}, \end{aligned} \quad (26)$$

of the environmental operators

$$B(t) = \sum_k g_k b_k e^{-i\omega_k t}. \quad (27)$$

These results hold for a generic environmental spectral density and the corresponding two-point correlation function. Taking, for example, a Lorentzian spectral density in resonance with the transition frequency of the qubit we find an exponential two-point correlation function

$$f(\tau) = \frac{1}{2} \gamma_0 \lambda e^{-\lambda|\tau|}, \quad (28)$$

where γ_0 describes the strength of the system-environment coupling and λ the spectral width which is related to the environmental correlation time by $\tau_E = \lambda^{-1}$. Solving Eq. (25) with this correlation function we find

$$G(t) = e^{-\lambda t/2} \left[\cosh\left(\frac{dt}{2}\right) + \frac{\lambda}{d} \sinh\left(\frac{dt}{2}\right) \right], \quad (29)$$

where $d = \sqrt{\lambda^2 - 2\gamma_0\lambda}$.

C. Completely positive semigroups

The simplest example of a quantum process is provided by a semigroup of completely positive dynamical maps, which is often considered as prototypical example of a quantum Markov process. In this case one assumes that the set (16) has the additional property

$$\Phi(t, 0) \Phi(s, 0) = \Phi(t+s, 0) \quad (30)$$

for all $t, s \geq 0$ and, hence, has the structure of a semigroup. Under very general mathematical conditions such a semigroup has an infinitesimal generator \mathcal{L} which allows us to write

$$\Phi(t, 0) = \exp[\mathcal{L}t]. \quad (31)$$

Accordingly, the reduced system state $\rho_S(t)$ obeys the master equation

$$\frac{d}{dt}\rho_S(t) = \mathcal{L}\rho_S(t). \quad (32)$$

The complete positivity of the semigroup leads to important statements on the general structure of the generator. The famous Gorini-Kossakowski-Sudarshan-Lindblad theorem [1, 2] states that \mathcal{L} is the generator of a semigroup of completely positive quantum dynamical maps if and only if it has the following form,

$$\mathcal{L}\rho_S = -i[H_S, \rho_S] + \sum_i \gamma_i \left[A_i \rho_S A_i^\dagger - \frac{1}{2} \{ A_i^\dagger A_i, \rho_S \} \right], \quad (33)$$

where H_S is a system Hamiltonian (which need not coincide with the system Hamiltonian H_S in the microscopic Hamiltonian (4)), the A_i are arbitrary system operators, often called Lindblad operators, describing the various decay modes of the system, and the γ_i are corresponding decay rates. This theorem has many far-reaching consequences and is extremely useful, in particular in phenomenological approaches since it guarantees a time evolution which is compatible with general physical principles for any master equation of the above structure. On the other hand, it is in general difficult to justify rigorously the assumption (30) and to derive a quantum master equation of the form (32) starting from a given system-environment model with a microscopic Hamiltonian (4). Such a derivation requires the validity of several approximations, the most important one being the so-called Markov approximation. This approximation presupposes a rapid decay of the two-point correlation functions of those environmental operators that describe the system-environment coupling H_I . More precisely, if τ_E describes the temporal width of these correlations and τ_R the relaxation or decoherence time of the system, the Markov approximation demands that

$$\tau_E \ll \tau_R. \quad (34)$$

This means that the environmental correlation time τ_E is small compared to the open system's relaxation or decoherence time τ_R , i.e., that we have a separation of time scales, the environmental variables being the fast and the system variables being the slow variables.

The Markov approximation is justified in many cases of physical interest. Typical examples of application are the weak coupling master equation, the quantum optical master equation describing the interaction of radiation with matter [3], and the master equation for a test particle in a quantum gas [15]. However, large couplings or interactions with low-temperature reservoirs can lead to strong correlations resulting in long memory times and in a failure of the Markov approximation. Moreover, the standard Markov condition (34) alone does *not* guarantee, in general, that the Markovian master equation provides a reasonable description of the dynamics, a situation which can occur, for example, for finite and/or structured reservoirs [16, 17].

D. Time-local master equations

There exists a whole bunch of different theoretical and numerical methods for the treatment of open quantum systems, beyond the assumption of a dynamical semigroup, such as projection operator techniques [18, 19], influence functional and path integral techniques [20], quantum Monte Carlo methods and stochastic wave function techniques [14, 21]. Here, we concentrate on a specific approach which is particularly suited for our purpose and which describes the open system dynamics in terms of a time-local master equation.

It is usually expected that the mathematical formulation of quantum processes describing effects of finite memory times in the system must necessarily involve equations of motion which are non-local in time. In fact, such a description is suggested by the Nakajima-Zwanzig projection operator technique which leads to an integro-differential equation for the reduced density matrix [18, 19]. However, even the presence of strong memory effects does not exclude the description of the dynamics in terms of a quantum master equation which is local in time, as may be seen from the following simple argument. According to Eq. (10) we have $\rho_S(t) = \Phi(t, 0)\rho_S(0)$. Assuming a smooth time-dependence we may differentiate this relation to get

$$\frac{d}{dt}\rho_S(t) = \dot{\Phi}(t, 0)\rho_S(0), \quad (35)$$

where the dot indicates the time derivative of $\Phi(t, 0)$. To obtain a local master equation we invert the relation (10), expressing $\rho_S(0)$ in terms of $\rho_S(t)$, which yields

$$\frac{d}{dt}\rho_S(t) = \dot{\Phi}(t, 0)\Phi^{-1}(t, 0)\rho_S(t). \quad (36)$$

Thus we see that the linear map $\mathcal{K}(t) = \dot{\Phi}(t, 0)\Phi^{-1}(t, 0)$ represents a time-dependent generator of the dynamics and we obtain a quantum master equation which is indeed local in time, providing a linear first-order differential equation for the open system state:

$$\frac{d}{dt}\rho_S(t) = \mathcal{K}(t)\rho_S(t). \quad (37)$$

We note that the above argument presupposes that the inverse $\Phi^{-1}(t, 0)$ of the map $\Phi(t, 0)$ exists. It is possible that the inverse of $\Phi(t, 0)$ and, hence, also the time-local generator $\mathcal{K}(t)$ do not exist. Such a situation can indeed occur for very strong system-environment couplings (see below). However, for an analytic time-dependence the inverse of $\Phi(t, 0)$ and the generator $\mathcal{K}(t)$ do exist apart from isolated singularities of the time axis [22]. It should also be emphasized that $\Phi^{-1}(t, 0)$ denotes the inverse of $\Phi(t, 0)$ regarded as linear map acting on the space of operators of the reduced system. The important point is that this does *not* imply that $\Phi^{-1}(t, 0)$ is required to be completely positive. In general, the inverse map is not only not completely positive, but even not positive. There exists a powerful method for the microscopic derivation of time-local master equations of the form (37) which is known as time-convolutionless projection operator technique [3, 23–27]. This technique yields a systematic expansion of the generator of the master equation in terms of ordered cumulants and many examples have been treated with this method [28–32].

The generator $\mathcal{K}(t)$ of the time-local master equation must of course preserve the Hermiticity and the trace, i.e., we have

$$[\mathcal{K}(t)A]^\dagger = \mathcal{K}(t)A^\dagger, \quad (38)$$

and

$$\text{tr}_S \{\mathcal{K}(t)A\} = 0. \quad (39)$$

From these requirements it follows that the generator must be of the following most general form,

$$\begin{aligned} \mathcal{K}(t)\rho_S = & -i[H_S(t), \rho_S] \\ & + \sum_i \gamma_i(t) \left[A_i(t)\rho_S A_i^\dagger(t) - \frac{1}{2} \left\{ A_i^\dagger(t)A_i(t), \rho_S \right\} \right]. \end{aligned} \quad (40)$$

We see that the structure of the generator provides a natural generalization of the Lindblad structure, in which the Hamiltonian $H_S(t)$, the Lindblad operators $A_i(t)$ as well as the various decay rates $\gamma_i(t)$ may dependent on time.

We briefly sketch the proof of (40) for a finite dimensional open system Hilbert space, $\dim \mathcal{H}_S = N_S$. To this end, we consider a fixed time t and a fixed complete system of operators F_i on \mathcal{H}_S , $i = 1, 2, \dots, N_S^2$, which are orthonormal with respect to the Hilbert-Schmidt scalar product,

$$\text{tr}_S\{F_i^\dagger F_j\} = \delta_{ij}. \quad (41)$$

Without loss of generality, we may further choose

$$F_{N_S^2} = \frac{1}{\sqrt{N_S}} I_S. \quad (42)$$

According to Lemma 2.3 of Ref. [1] any linear map $\mathcal{K}(t)$ satisfying Eqs. (38) and (39) can then be written as

$$\begin{aligned} \mathcal{K}(t)\rho_S = & -i[H_S(t), \rho_S] \\ & + \sum_{i,j=1}^{N_S^2-1} c_{ij}(t) \left[F_i \rho_S F_j^\dagger - \frac{1}{2} \{F_j^\dagger F_i, \rho_S\} \right] \end{aligned} \quad (43)$$

with a self-adjoint operator $H_S(t)$ and a Hermitian matrix $c(t) = (c_{ij}(t))$. Diagonalizing this matrix with the help of a unitary matrix $u(t) = (u_{ij}(t))$,

$$u(t)c(t)u^\dagger(t) = \text{diag}(\gamma_1(t), \gamma_2(t), \dots, \gamma_{N_S^2-1}(t)), \quad (44)$$

and introducing the new operators

$$A_i(t) = \sum_{j=1}^{N_S^2-1} u_{ij}^*(t) F_j, \quad (45)$$

one obtains the form (40) of the generator.

The structure (40) takes into account the Hermiticity and trace preservation of the dynamics, but does not guarantee its complete positivity. The formulation of necessary and sufficient conditions for the complete positivity of the dynamics of this generator is an important unsolved problem. However, in the case that the rates are positive for all times,

$$\gamma_i(t) \geq 0, \quad (46)$$

the resulting dynamics is indeed completely positive, since the generator is then in Lindblad form for each fixed $t \geq 0$.

As an example let us consider the dynamical map given by Eqs. (21)-(24). In this case the time-local generator takes the form [14]

$$\begin{aligned} \mathcal{K}(t)\rho_S = & -\frac{i}{2}S(t)[\sigma_+\sigma_-, \rho_S] \\ & +\gamma(t)\left[\sigma_-\rho_S\sigma_+ - \frac{1}{2}\{\sigma_+\sigma_-, \rho_S\}\right], \end{aligned} \quad (47)$$

where we have introduced the definitions

$$\gamma(t) = -2\Re\left(\frac{\dot{G}(t)}{G(t)}\right), \quad S(t) = -2\Im\left(\frac{\dot{G}(t)}{G(t)}\right). \quad (48)$$

With this generator Eq. (37) represents an exact master equation of the model. The quantity $S(t)$ plays the role of a time-dependent frequency shift, and $\gamma(t)$ can be interpreted as a time-dependent decay rate. Due to the time dependence of these quantities the process does generally not represent a dynamical semigroup, of course. Note that the generator is finite as long as $G(t) \neq 0$; at the zeros of $G(t)$ the inverse of the dynamical map $\Phi(t, 0)$ does not exist. An example is provided by the zeros of the function of Eq. (29) which appear in the strong coupling regime $\gamma_0 > \lambda/2$.

We can also see explicitly how the standard Markov limit arises in this model. Considering the particular case (29), we observe that in the limit of small $\alpha = \gamma_0/\lambda$ we may approximate $G(t) \approx e^{-\gamma_0 t/2}$. This approximation can also be obtained directly from Eq. (25) by replacing the two-point correlation function $f(t - t_1)$ with the delta-function $\gamma_0\delta(t - t_1)$, which is conventionally regarded as the Markovian limit. Equation (48) then yields $S(t) = 0$ and $\gamma(t) = \gamma_0$, i.e., the generator (47) assumes the form of a Lindblad generator of a quantum dynamical semigroup. The quantity α can also be written as the ratio of the environmental correlations time $\tau_E = \lambda^{-1}$ and the relaxation time $\tau_R = \gamma_0^{-1}$ of the system:

$$\alpha = \frac{\tau_E}{\tau_R}. \quad (49)$$

Thus we see that the standard Markov condition (34) indeed leads to a Markovian semigroup here. We also mention that the time-convolutionless projection operator technique yields an expansion of the generator (47) in powers of this ratio α [3].

E. Divisibility of dynamical maps

A family of dynamical maps $\Phi(t, 0)$ is defined to be divisible if for all $t_2 \geq t_1 \geq 0$ there exists a CPT map $\Phi(t_2, t_1)$ such that the relation

$$\Phi(t_2, 0) = \Phi(t_2, t_1)\Phi(t_1, 0). \quad (50)$$

holds. Note that in this equation the left-hand side as well as the second factor on the right-hand side are fixed by the given family of dynamical maps. Hence, Eq. (50) requires the existence of a certain linear transformation $\Phi(t_2, t_1)$ which maps the states at time t_1 to the states at time t_2 and represents a CPT map (which may be a unitary transformation). This definition differs slightly from the usual definition for the divisibility of a CPT map Λ , according to which Λ is said to be divisible if there exist CPT maps Λ_1 and Λ_2 such that $\Lambda = \Lambda_1\Lambda_2$, where one requires that neither Λ_1 nor Λ_2 is a unitary transformation, for otherwise the statement is trivial [33]. There are many quantum processes which are not divisible. For instance, if $\Phi(t_1, 0)$ is not invertible, a linear map $\Phi(t_2, t_1)$ which fulfills Eq. (50) may not exist. Moreover, even if a linear map $\Phi(t_2, t_1)$ satisfying Eq. (50) does exist, this map needs not be completely positive, and not even positive.

The simplest example of a divisible quantum process is given by a dynamical semigroup. In fact, for a semigroup we have $\Phi(t, 0) = \exp[\mathcal{L}t]$ and, hence, Eq. (50) is trivially satisfied with the CPT map $\Phi(t_2, t_1) = \exp[\mathcal{L}(t_2 - t_1)]$.

Consider now a quantum process given by the time-local master equation (37) with a time dependent generator (40). The dynamical maps can then be represented in terms of a time-ordered exponential,

$$\Phi(t, 0) = \text{T exp} \left[\int_0^t dt' \mathcal{K}(t') \right], \quad t \geq 0, \quad (51)$$

where T denotes the chronological time-ordering operator. We can also define the maps

$$\Phi(t_2, t_1) = \text{T exp} \left[\int_{t_1}^{t_2} dt' \mathcal{K}(t') \right], \quad t_2 \geq t_1 \geq 0, \quad (52)$$

such that the composition law $\Phi(t_2, 0) = \Phi(t_2, t_1)\Phi(t_1, 0)$ holds by construction. The maps $\Phi(t_2, t_1)$ are completely positive, as is required by the divisibility condition (50), if and only if the rates $\gamma_i(t)$ of the generator (40) are positive functions. Thus we see that divisibility is equivalent to positive rates in the time-local master equation [7]. To prove this statement

suppose first that $\gamma_i(t) \geq 0$. The generator $\mathcal{K}(t)$ is then in the Lindblad form for each fixed $t \geq 0$ and, therefore, Eq. (52) represents completely positive maps. Conversely, assume that the maps defined by Eq. (52) are completely positive. It follows from this equation that the generator is given by

$$\mathcal{K}(t) = \left. \frac{d}{d\tau} \right|_{\tau=0} \Phi(t + \tau, t). \quad (53)$$

Since $\Phi(t + \tau, t)$ is completely positive for all $t, \tau \geq 0$ and satisfies $\Phi(t, t) = I$, this generator must be in Lindblad form for each fixed t , i.e., it must have the form (40) with $\gamma_i(t) \geq 0$.

The dynamical map $\Phi(t, 0)$ given by Eqs. (21)-(24) is completely positive if and only if $|G(t)| \leq 1$. It is easy to verify that $\Phi(t, 0)$ can be decomposed as in Eq. (50), where the map $\Phi(t_2, t_1)$ is given by [7]

$$\begin{aligned} \rho_{11}(t_2) &= \left| \frac{G(t_2)}{G(t_1)} \right|^2 \rho_{11}(t_1), \\ \rho_{00}(t_2) &= \rho_{00}(t_1) + \left(1 - \left| \frac{G(t_2)}{G(t_1)} \right|^2 \right) \rho_{11}(t_1), \\ \rho_{10}(t_2) &= \frac{G(t_2)}{G(t_1)} \rho_{10}(t_1), \\ \rho_{01}(t_2) &= \frac{G^*(t_2)}{G^*(t_1)} \rho_{01}(t_1). \end{aligned} \quad (54)$$

It follows from these equations that a necessary and sufficient condition for the complete positivity of $\Phi(t_2, t_1)$ is given by

$$|G(t_2)| \leq |G(t_1)|. \quad (55)$$

Thus we see that the dynamical map of the model is divisible if and only if $|G(t)|$ is a monotonically decreasing function of time. Note that this statement holds true also for the case that $G(t)$ vanishes at some finite time. The rate $\gamma(t)$ in Eq. (48) can be written as

$$\gamma(t) = -\frac{2}{|G(t)|} \frac{d}{dt} |G(t)|. \quad (56)$$

This shows that any increase of $|G(t)|$ leads to a negative decay rate in the corresponding generator (47), and illustrates the equivalence of the non-divisibility of the dynamical map and the occurrence of a temporarily negative rate in the time-local master equation demonstrated above.

III. INFORMATION FLOW AND NON-MARKOVIAN QUANTUM DYNAMICS

In the classical theory of stochastic processes a Markov process is defined by the Markov condition, which is a condition for the hierarchy of the n -point probability distribution functions pertaining to the process. Since such a hierarchy does not exist in quantum mechanics the question arises, how do quantum memory effects manifest themselves in the dynamical behavior of complex open quantum systems and how one can rigorously define and quantify non-Markovianity in the quantum case? Of course, such a definition must be independent of the specific mathematical representation of the open system's dynamics, i.e., it must be formulated completely by means of the quantum dynamical map of the process. In Ref. [6] two different concepts of non-Markovianity have been proposed. The first one uses the fact that local quantum operations can never increase the entanglement between the open system and an isomorphic auxiliary system. By employing an appropriate measure for entanglement, a given dynamical evolution is then defined to be non-Markovian if a temporary increase of the entanglement measure takes place, and the size of this increase provides a measure for the degree of non-Markovianity. Within the second concept a quantum process is defined to be non-Markovian if and only if the dynamical map $\Phi(t, 0)$ is non-divisible. The corresponding measure for non-Markovianity quantifies the amount to which the dynamical map violates the divisibility condition, thus representing a measure for the non-divisibility character of the process.

Here, we discuss a recent approach which defines and quantifies the emergence of quantum memory effects entirely in terms of the exchange of information between the open system and its environment [5]. In order to quantify this exchange of information we will use the trace distance as a measure for the distance between quantum states.

A. Trace distance and distinguishability of quantum states

The trace norm of a trace class operator A is defined by

$$||A|| = \text{tr}|A| = \text{tr}\sqrt{A^\dagger A}. \quad (57)$$

If A is self-adjoint with eigenvalues a_i , this formula reduces to

$$||A|| = \sum_i |a_i|. \quad (58)$$

Hence, the trace norm of a self-adjoint operator is equal to the sum of the moduli of its eigenvalues (counting their multiplicities). The trace norm leads to a natural and useful measure for the distance between two quantum states represented by positive operators ρ^1 and ρ^2 with unit trace, which is known as the trace distance:

$$D(\rho^1, \rho^2) = \frac{1}{2} \|\rho^1 - \rho^2\| = \frac{1}{2} \text{tr} |\rho^1 - \rho^2|. \quad (59)$$

The trace distance is well defined and finite for all pairs of quantum states and provides a metric on the space $S(\mathcal{H})$ of physical states. We list some of the most important mathematical properties of this metric which will be needed later on (most of the proofs may be found in Refs. [10, 11]).

1. The trace distance between any pair of states satisfies

$$0 \leq D(\rho^1, \rho^2) \leq 1. \quad (60)$$

Of course, we have $D(\rho^1, \rho^2) = 0$ if and only if $\rho^1 = \rho^2$, while the upper bound is reached, i.e., $D(\rho^1, \rho^2) = 1$ if and only if ρ^1 and ρ^2 are orthogonal, which means that the supports of ρ^1 and ρ^2 are orthogonal. (The support is defined to be the space spanned by the eigenstates with nonzero eigenvalue.) Moreover, $D(\rho^1, \rho^2)$ is obviously symmetric in the input arguments, and satisfies the triangular inequality:

$$D(\rho^1, \rho^2) \leq D(\rho^1, \rho^3) + D(\rho^3, \rho^2). \quad (61)$$

2. If $\rho^1 = |\psi^1\rangle\langle\psi^1|$ and $\rho^2 = |\psi^2\rangle\langle\psi^2|$ are pure states the following explicit formula for the trace distance can easily be derived,

$$D(\rho^1, \rho^2) = \sqrt{1 - |\langle\psi^1|\psi^2\rangle|^2}. \quad (62)$$

If the underlying Hilbert space is two-dimensional (qubit), spanned by basis states $|1\rangle$ and $|0\rangle$, the trace distance between two states with matrix elements ρ_{ij}^1 and ρ_{ij}^2 is found to be

$$D(\rho^1, \rho^2) = \sqrt{a^2 + |b|^2}, \quad (63)$$

where $a = \rho_{11}^1 - \rho_{11}^2$ is the difference of the populations, and $b = \rho_{10}^1 - \rho_{10}^2$ is the difference of the coherences of the two states.

3. The trace distance is sub-additive with respect to tensor products of states which means that

$$D(\rho^1 \otimes \sigma^1, \rho^2 \otimes \sigma^2) \leq D(\rho^1, \rho^2) + D(\sigma^1, \sigma^2). \quad (64)$$

In addition we have

$$D(\rho^1 \otimes \sigma, \rho^2 \otimes \sigma) = D(\rho^1, \rho^2). \quad (65)$$

4. The trace distance is invariant under unitary transformations U ,

$$D(U\rho^1 U^\dagger, U\rho^2 U^\dagger) = D(\rho^1, \rho^2). \quad (66)$$

More generally, all trace preserving and completely positive maps, i.e., all trace preserving quantum operations Λ are contractions of the trace distance,

$$D(\Lambda\rho^1, \Lambda\rho^2) \leq D(\rho^1, \rho^2). \quad (67)$$

Note that the condition of the trace preservation is important here, and that this inequality also holds for the larger class of trace preserving positive maps [34].

5. The trace distance can be represented as the maximum of a certain functional:

$$D(\rho^1, \rho^2) = \max_{\Pi} \text{tr} \{ \Pi (\rho^1 - \rho^2) \}. \quad (68)$$

The maximum is taken over all projection operators Π . Alternatively, one can take the maximum over all positive operators A with $A \leq I$. Note that this formula is symmetric, i.e., we can also write $D(\rho^1, \rho^2) = \max_{\Pi} \text{tr} \{ \Pi (\rho^2 - \rho^1) \}$ where, however, the maximum is then assumed for a different projection Π .

The trace distance between two quantum states ρ^1 and ρ^2 has a direct physical interpretation which is based on the representation (68). Consider two parties, Alice and Bob. Alice prepares a quantum system in one of two states ρ^1 or ρ^2 with probability $\frac{1}{2}$ each, and then sends the system to Bob. It is Bob's task to find out by a single measurement on the system whether the system state was ρ^1 or ρ^2 . It turns out that Bob cannot always distinguish the states with certainty, but there is an optimal strategy which allows him to achieve the maximal possible success probability given by

$$P_{\max} = \frac{1}{2} [1 + D(\rho^1, \rho^2)]. \quad (69)$$

Thus we see that the trace distance represents the bias in favor of the correct state identification which Bob can achieve through an optimal strategy. The trace distance $D(\rho^1, \rho^2)$ can therefore be interpreted as a measure for the distinguishability of the quantum states ρ^1 and ρ^2 [8–10].

According to Eq. (68) Bob's optimal strategy consists in measuring the projection Π for which the maximum in this relation is assumed, and in associating the outcome $\Pi = 1$ with the state ρ^1 , and the outcome $\Pi = 0$ with the state ρ^2 . Under the condition that the system state was ρ^1 he then has correctly identified this state with probability $\text{tr}\{\Pi\rho^1\}$, while under the condition that the system state was ρ^2 his answer is correct with probability $\text{tr}\{(I - \Pi)\rho^2\}$. Since both possibilities occur with a probability of $\frac{1}{2}$ we obtain the success probability

$$\begin{aligned} P_{\max} &= \frac{1}{2}\text{tr}\{\Pi\rho^1\} + \frac{1}{2}\text{tr}\{(I - \Pi)\rho^2\} \\ &= \frac{1}{2} [1 + \text{tr}\{\Pi(\rho^1 - \rho^2)\}] \\ &= \frac{1}{2} [1 + D(\rho^1, \rho^2)], \end{aligned} \tag{70}$$

which proves Eq. (69). We further see that a state identification with certainty, $P_{\max} = 1$, can be achieved if and only if $D(\rho^1, \rho^2) = 1$, i.e., if and only if ρ^1 and ρ^2 are orthogonal, in which case Bob's optimal strategy is to measure the projection Π onto the support of ρ^1 (or of ρ^2).

B. Definition of non-Markovian quantum dynamics

As we have seen in Sec. III A the trace distance $D(\rho^1, \rho^2)$ between two quantum states ρ^1 and ρ^2 can be interpreted as the distinguishability of these states. An important conclusion from this interpretation is that according to the contraction property of Eq. (67) no completely positive and trace preserving quantum operation can increase the distinguishability between quantum states. Consider two initial states $\rho_S^1(0)$ and $\rho_S^2(0)$ of an open system and the corresponding time evolved states

$$\rho_S^{1,2}(t) = \Phi(t, 0)\rho_S^{1,2}(0). \tag{71}$$

Since the dynamical maps $\Phi(t, 0)$ are completely positive and trace preserving, the trace distance between the time evolved states can never be larger than the trace distance between

the initial states,

$$D(\rho_S^1(t), \rho_S^2(t)) \leq D(\rho_S^1(0), \rho_S^2(0)). \quad (72)$$

Thus, no quantum process describable by a family of CPT dynamical maps can ever increase the distinguishability of a pair of states over its initial value. Of course, this general feature does not imply that $D(\rho_S^1(t), \rho_S^2(t))$ is a monotonically decreasing function of time.

We can interpret the dynamical change of the distinguishability of the states of an open system in terms of a flow of information between the system and its environment. When a quantum process reduces the distinguishability of states, information is flowing from the system to the environment. Correspondingly, an increase of the distinguishability signifies that information flows from the environment back to the system. The invariance under unitary transformations (66) indicates that information is preserved under the dynamics of closed systems. On the other hand, the contraction property of Eq. (67) guarantees that the maximal amount of information the system can recover from the environment is the amount of information earlier flowed out the system.

Our definition for quantum non-Markovianity is based on the idea that for Markovian processes any two quantum states become less and less distinguishable under the dynamics, leading to a perpetual loss of information into the environment. Quantum memory effect thus arise if there is a temporal flow of information from the environment to the system. The information flowing back from the environment allows the earlier open system states to have an effect on the later dynamics of the system, which implies the emergence of memory effects [5].

In view of these considerations, we thus define a quantum process described in terms of a family of quantum dynamical maps $\Phi(t, 0)$ to be non-Markovian if and only if there is a pair of initial states $\rho_S^{1,2}(0)$ such that the trace distance between the corresponding states $\rho_S^{1,2}(t)$ increases at a certain time $t > 0$:

$$\sigma(t, \rho_S^{1,2}(0)) \equiv \frac{d}{dt} D(\rho_S^1(t), \rho_S^2(t)) > 0, \quad (73)$$

where $\sigma(t, \rho_S^{1,2}(0))$ denotes the rate of change of the trace distance at time t corresponding to the initial pair of states.

This definition for quantum non-Markovianity has many important consequences for the general classification of quantum processes. In particular, it implies that all divisible families of dynamical maps are Markovian, including of course the class of quantum dynamical

semigroups. To prove this statement suppose that $\Phi(t, 0)$ is divisible. For any pair of initial states $\rho_S^{1,2}(0)$ we then have

$$\rho_S^{1,2}(t + \tau) = \Phi(t + \tau, t)\rho_S^{1,2}(t), \quad t, \tau \geq 0. \quad (74)$$

Since $\Phi(t + \tau, t)$ is a CPT map we can apply the contraction property (67) to obtain:

$$D(\rho_S^1(t + \tau), \rho_S^2(t + \tau)) \leq D(\rho_S^1(t), \rho_S^2(t)). \quad (75)$$

This shows that for all divisible dynamical maps the trace distance decreases monotonically and that, therefore, these processes are Markovian.

Hence, we see that non-Markovian quantum processes must necessarily be described by non-divisible dynamical maps and by time-local master equations whose generator (40) involves at least one temporarily negative rate $\gamma_i(t)$. However, it is important to note that the converse of this statement is not true, i.e., there exist non-divisible dynamical processes which are Markovian in the sense of the above definition. For such processes the effect of the contribution of the decay channels with a negative rate is overcompensated by the channels with a positive decay rate, resulting in a total information flow directed from the open system to the environment. Further implications are discussed in [35], and specific examples for this class of processes are constructed in [36, 37].

We have interpreted the change of the distinguishability of quantum states as arising from an exchange of information between the open system and its environment. If the distinguishability decreases, for example, we say that information is flowing from the open system into the environment. The more precise meaning of this statement is that the available relative information on the considered pair of states, that is, the information which enables one to distinguish these states is lost from the open system, i.e., when only measurements on the open system's degrees of freedom can be performed. This does not imply that the information lost is now contained completely in the reduced states of the environment, but instead, this information can also be carried by the system-environment correlations. And vice versa, if we say that information flows from the environment back to the system, this means that the distinguishability of the open system states increases because information is regained by the open system, information which was carried by the environmental degrees of freedom or by the correlations between the degrees of freedom of system and environment (see also Sec. IV).

C. Construction of a measure for the degree of non-Markovianity

For a non-Markovian process described by a family of dynamical maps $\Phi(t, 0)$ information must flow from the environment to the system for some interval of time and thus we must have $\sigma > 0$ for this time interval. A measure of non-Markovianity should measure the total increase of the distinguishability over the whole time evolution, i.e., the total amount of information flowing from the environment back to the system. This suggests defining a measure $\mathcal{N}(\Phi)$ for the non-Markovianity of a quantum process through [5]

$$\mathcal{N}(\Phi) = \max_{\rho_S^{1,2}(0)} \int_{\sigma>0} dt \sigma(t, \rho_S^{1,2}(0)). \quad (76)$$

The time integration is extended over all time intervals (a_i, b_i) in which σ is positive and the maximum is taken over all pairs of initial states. Due to Eq. (73) the measure can be written as

$$\mathcal{N}(\Phi) = \max_{\rho^{1,2}(0)} \sum_i [D(\rho_S^1(b_i), \rho_S^2(b_i)) - D(\rho_S^1(a_i), \rho_S^2(a_i))]. \quad (77)$$

To calculate this quantity one first determines for any pair of initial states the total growth of the trace distance over each time interval (a_i, b_i) and sums up the contribution of all intervals. $\mathcal{N}(\Phi)$ is then obtained by determining the maximum over all pairs of initial states.

To illustrate our definition of non-Markovianity and the measure (76) we again refer to the example of the dynamical map given by Eqs. (21)-(24). According to Eq. (63) the time evolution of the trace distance corresponding to any pair of initial states $\rho_S^1(0)$ and $\rho_S^2(0)$ is given by

$$D(\rho_S^1(t), \rho_S^2(t)) = |G(t)| \sqrt{|G(t)|^2 a^2 + |b|^2}, \quad (78)$$

where $a = \rho_{11}^1(0) - \rho_{11}^2(0)$ and $b = \rho_{10}^1(0) - \rho_{10}^2(0)$. The time derivative of this expression yields

$$\sigma(t, \rho_S^{1,2}(0)) = \frac{2|G(t)|^2 a^2 + |b|^2}{\sqrt{|G(t)|^2 a^2 + |b|^2}} \frac{d}{dt} |G(t)|. \quad (79)$$

We conclude from this equation that the trace distance increases at time t if and only if the function $|G(t)|$ increases at this point of time. It follows that the process is non-Markovian, $\mathcal{N}(\Phi) > 0$, if and only if the dynamical map is non-divisible, which in turn is equivalent to a temporarily negative rate $\gamma(t)$ in the time-local master equation with generator (47).

Consider, for example, the case of an exponential correlation function which leads to the expression (29) for the function $G(t)$. For small couplings, $\gamma_0 < \lambda/2$, this function decreases monotonically. The dynamical map is then divisible, the rate $\gamma(t)$ is always positive, and the process is Markovian. However, in the strong coupling regime, $\gamma_0 > \lambda/2$, the function $|G(t)|$ starts to oscillate, showing a non-monotonic behavior. Consequently, the dynamical map is then no longer divisible and the process is non-Markovian. Note that in the strong coupling regime the rate $\gamma(t)$ diverges at the zeros of $G(t)$. However, the master equation can still be used to describe the evolution between successive zeros and, therefore, the connection between non-Markovianity, divisibility and a negative rate in the master equation remains valid. There is thus a threshold $\gamma_0 = \lambda/2$ for the system-reservoir coupling below which $\mathcal{N}(\Phi) = 0$. One finds, as expected, that for $\gamma_0 > \lambda/2$ the non-Markovianity increases monotonically with increasing system-environment coupling [7]. Moreover, it is easy to show with the help of Eq. (78) that the maximum in Eq. (76) is attained for $a = 0$ and $|b| = 1$ [38]. This means that the optimal pairs of initial states correspond to antipodal points on the equator of the Bloch sphere representing the qubit.

The non-Markovianity measure (76) has been employed recently for the theoretical characterization and quantification of memory effects in various physical systems. An application to the information flow in the energy transfer dynamics of photosynthetic complexes has been developed in [39], and to ultracold atomic gases in [40]. Further applications to memory effects in the dynamics of qubits coupled to spin chains [41] and to complex quantum systems with regular and chaotic dynamics [42] have been reported. A series of examples and details of the relation between the classical and the quantum concepts of non-Markovianity are discussed in [43].

The non-Markovianity measure (76) represents a novel experimentally accessible quantity. Quite recently, two experiments determining this quantity have been carried out employing photons moving in optical fibers or birefringent materials [44, 45]. In these experiments the open system is provided by the polarization degrees of freedom of the photons, while the environment is given by their frequency (mode) degrees of freedom. It has been demonstrated that a complete determination of the measure is experimentally possible, including the realization of the maximization over initial states in the definition (76). It was shown further that a careful preparation of the initial environmental state allows to control the information flow between the system and its environment and to observe the transition from

the Markovian to the non-Markovian regime through quantum state tomography carried out on the open system. This means that an experimental quantification and control of memory effects in open quantum systems is indeed feasible, which could be useful in the development of quantum memory and communication devices.

IV. THE ROLE OF INITIAL SYSTEM-ENVIRONMENT CORRELATIONS

As we have seen in Sec. II B the construction of a completely positive quantum dynamical map $\Phi(t, 0)$ acting on the open system's state space is based on the assumption of an uncorrelated total initial state. In this section we will derive general inequalities which express the role of initial system-environment correlations in the subsequent dynamical evolution and discuss suitable strategies for the local detection of such correlations [46].

A. General relations describing the effect of initial correlations

While in the general case of correlated initial states a quantum dynamical map can only be defined for a restricted set of states [47], we can of course always introduce a linear map

$$\Lambda(t, 0) : S(\mathcal{H}_S \otimes \mathcal{H}_E) \longrightarrow S(\mathcal{H}_S) \quad (80)$$

on the total system's state space $S(\mathcal{H}_S \otimes \mathcal{H}_E)$ by means of

$$\rho(0) \mapsto \rho_S(t) = \Lambda(t, 0)\rho(0) = \text{tr}_E \{U(t)\rho(0)U^\dagger(t)\}. \quad (81)$$

This map takes any initial state $\rho(0)$ of the total system to the corresponding reduced open system state $\rho_S(t)$ at time t . Since unitary transformations and partial traces are CPT maps, the composite maps $\Lambda(t, 0)$ are again CPT maps. Considering a pair of total initial state $\rho^{1,2}(0)$ and the corresponding open system states at time t ,

$$\rho_S^{1,2}(t) = \Lambda(t, 0)\rho^{1,2}(0), \quad (82)$$

we then have by use of the contraction property (67) for CPT maps:

$$D(\rho_S^1(t), \rho_S^2(t)) \leq D(\rho^1(0), \rho^2(0)), \quad (83)$$

which means that the distinguishability of the open system states can never be larger than the distinguishability of the total initial states. Subtracting the initial trace distance of the

open system states we obtain:

$$\begin{aligned} D(\rho_S^1(t), \rho_S^2(t)) - D(\rho_S^1(0), \rho_S^2(0)) \\ \leq I(\rho^1(0), \rho^2(0)) \equiv D(\rho^1(0), \rho^2(0)) - D(\rho_S^1(0), \rho_S^2(0)). \end{aligned} \quad (84)$$

The increase of the trace distance between the open system states is thus bounded from above by the quantity $I(\rho^1(0), \rho^2(0))$ on the right-hand side of this equation. This quantity represents the distinguishability of the total initial states minus the distinguishability of the initial open system states. It can be interpreted as the information on the total system states which is outside the open system, i.e., which is inaccessible through local measurements on the open system. Thus we see that the increase of the distinguishability of the open system states and, hence, the flow of information to the open system is bounded by the information which is outside the open system at the initial time.

We now show how the upper bound of Eq. (84) is connected to the correlations in the initial states. To this end, we use twice the triangular inequality (61) to get:

$$\begin{aligned} D(\rho^1(0), \rho^2(0)) &\leq D(\rho^1(0), \rho_S^1(0) \otimes \rho_E^1(0)) + D(\rho^2(0), \rho_S^1(0) \otimes \rho_E^1(0)) \\ &\leq D(\rho^1(0), \rho_S^1(0) \otimes \rho_E^1(0)) + D(\rho^2(0), \rho_S^2(0) \otimes \rho_E^2(0)) \\ &\quad + D(\rho_S^1(0) \otimes \rho_E^1(0), \rho_S^2(0) \otimes \rho_E^2(0)). \end{aligned} \quad (85)$$

With the help of the subadditivity (64) of the trace distance we find

$$D(\rho_S^1(0) \otimes \rho_E^1(0), \rho_S^2(0) \otimes \rho_E^2(0)) \leq D(\rho_S^1(0), \rho_S^2(0)) + D(\rho_E^1(0), \rho_E^2(0)). \quad (86)$$

Combining this with Eqs. (85) and (84) we finally obtain

$$\begin{aligned} D(\rho_S^1(t), \rho_S^2(t)) - D(\rho_S^1(0), \rho_S^2(0)) &\leq D(\rho^1(0), \rho_S^1(0) \otimes \rho_E^1(0)) + D(\rho^2(0), \rho_S^2(0) \otimes \rho_E^2(0)) \\ &\quad + D(\rho_S^1(0), \rho_S^2(0)). \end{aligned} \quad (87)$$

For any total system state ρ the trace distance $D(\rho, \rho_S \otimes \rho_E)$ between ρ and the product of its marginals $\rho_S \otimes \rho_E$ represents a measure for the correlations in the state ρ , which quantifies how well ρ can be distinguished from the corresponding product state $\rho_S \otimes \rho_E$. Thus, Eq. (87) demonstrates that a dynamical increase of the trace distance of the open system states over the initial value,

$$D(\rho_S^1(t), \rho_S^2(t)) - D(\rho_S^1(0), \rho_S^2(0)) > 0, \quad (88)$$

implies that the corresponding initial environmental states $\rho_E^{1,2}(0)$ are different or that at least one of the total initial states $\rho^{1,2}(0)$ is correlated.

As an example we consider the special case of an initial pair of states given by a correlated state $\rho^1(0)$ and the uncorrelated product of its marginals $\rho^2(0) = \rho_S^1(0) \otimes \rho_E^1(0)$. Hence, we have $\rho_S^1(0) = \rho_S^2(0)$ and $\rho_E^1(0) = \rho_E^2(0)$, and Eq. (87) reduces to

$$D(\rho_S^1(t), \rho_S^2(t)) \leq D(\rho^1(0), \rho_S^1(0) \otimes \rho_E^1(0)). \quad (89)$$

Thus, the increase of the trace distance between $\rho_S^1(t)$ and $\rho_S^2(t)$ (which is zero initially) is bounded from above by the amount of correlations in the initial state $\rho^1(0)$.

We further remark that in the case of two uncorrelated states with the same environmental state, i.e., $\rho^{1,2}(0) = \rho_S^{1,2}(0) \otimes \rho_E(0)$, the right-hand side of the inequality (87) vanishes, and we are led again to the contraction property (72) for CPT dynamical maps on the reduced state space. Thus, Eq. (87) represents a generalization of the contraction property of dynamical maps on the reduced state space.

B. Witnessing system-environment correlations by local operations

Equation (87) leads to an experimentally realizable scheme for the local detection of correlations in an unknown total system's initial state $\rho^1(0)$ as follows [46]. First, one generates a second reference state $\rho^2(0)$ by applying a local trace-preserving quantum operation \mathcal{E} ,

$$\rho^2(0) = (\mathcal{E} \otimes I)\rho^1(0). \quad (90)$$

The operation \mathcal{E} acts only on the variables of the open system, and can be realized, for example, by the measurement of an observable of the open system, or by a unitary transformation induced, e.g., through an external control field. It is easy to check that $\rho^1(0)$ and $\rho^2(0)$ lead to the same reduced environmental state, i.e., we have $\rho_E^1(0) = \rho_E^2(0)$ and, hence, Eq. (87) yields

$$D(\rho_S^1(t), \rho_S^2(t)) - D(\rho_S^1(0), \rho_S^2(0)) \leq D(\rho^1(0), \rho_S^1(0) \otimes \rho_E^1(0)) + D(\rho^2(0), \rho_S^2(0) \otimes \rho_E^2(0)). \quad (91)$$

This inequality shows that any dynamical increase of the trace distance between the open system states over the initial value implies the presence of correlations in the initial state $\rho^1(0)$. In fact, if one finds that the left-hand side of the inequality is greater than zero, then

either $\rho^1(0)$ or $\rho^2(0)$ must be correlated. If $\rho^1(0)$ was uncorrelated, then also $\rho^2(0)$ must be uncorrelated since it is obtained from $\rho^1(0)$ through application of a local operation. Thus, any increase of the trace distance of the open system states over the initial value witnesses correlations in $\rho^1(0)$.

We note that this strategy for the local detection of initial correlations requires only local control and measurements of the open quantum system. It neither demands knowledge about the structure of the environment or of the system-environment interaction, nor a full knowledge of the initial system-environment state $\rho^1(0)$. Moreover, there is no principal restriction on the operation \mathcal{E} used to generate the second state $\rho^2(0)$, which makes the scheme very flexible in practice. In fact, experimental realizations of the scheme have been reported recently [48, 49]. Further examples and applications to the study of correlations in thermal equilibrium states are discussed in Ref. [50]. Moreover, by taking \mathcal{E} to be a pure dephasing operation the scheme enables the local detection of nonclassical correlations, i.e., of total initial states with nonzero quantum discord [51]. This fact has been shown very recently in Ref. [52] where also a statistical approach to initial correlations on the basis of random matrix theory has been developed.

V. CONCLUSIONS

We have discussed a definition for the non-Markovianity of quantum processes in open systems and developed a corresponding measures for the size of quantum memory effects. Our considerations are based on the quantification of the information flow between the open system and its environment in terms of the trace distance between quantum states of the open system. The great advantage of this distance measure is the fact that it admits a natural and clear physical interpretation as the distinguishability of the states through local measurements carried out on the open system. According to our definition the key feature of quantum non-Markovianity is the temporal increase of the distinguishability which can be interpreted as a backflow of information from the environment to the open system. As we have seen this concept allows a natural extension to the case of initial system-environment correlations. While a quantum dynamical map acting on the open system's state space does in general not exist in this case, the trace distance between pairs of states of the open system leads to a dynamical witness for the presence of initial correlations in the total system state.

Given that the initial correlations were created in the past from a product state through a system-environment interaction, the increase of the trace distance of the open system states over its initial value signifies that the open system regains information which was lost previously.

There are of course several alternatives and possible modifications of the quantum measure for non-Markovianity studied here. One possibility is to employ alternative distance measures for quantum states under which trace preserving quantum operations are contractive, such as the relative entropy or the Bures distance which is based on the fidelity [10]. In particular, the relative entropy represents a possibility which is natural both from an information theoretic perspective and from the point of view of nonequilibrium thermodynamics, since the negative rate of change of the entropy relative to an invariant thermal equilibrium state can be interpreted as entropy production [53, 54]. However, a disadvantage of the relative entropy is given by the fact that it is often infinite, leading to singularities of the measure [5]. In several cases, in particular for infinite dimensional Hilbert spaces, the determination of the trace distance could be an extremely difficult task. An analytical formula for the trace distance is not even known for Gaussian quantum states. It seems that in those cases it is much easier to work with the Bures distance or the fidelity which also leads to useful lower and upper bounds for the trace distance [55]. The Hilbert-Schmidt distance which is technically much easier to deal with cannot be used for the quantification of non-Markovianity because trace preserving quantum operations are, in general, not contractive for this metric [56]. A further possibility is to define the measure for quantum non-Markovianity by means of alternative functionals of the family of dynamical maps which quantify the dynamical increase of the chosen distance measure for quantum states.

Both the quantum measure for non-Markovianity and the witness for initial system-environment correlations studied here have been demonstrated very recently to be experimentally measurable quantities [44, 45, 48, 49]. These experiments have paved the way for a series of further investigations of quantum memory effects in composite, multipartite open systems. A central goal of the quantum theory of open system is thus the design of appropriate schemes and models for their theoretical treatment and, in particular, the development of efficient numerical methods for the determination of quantum measures for non-Markovianity.

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